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A Note on the Mean Convergence of Lagrange Interpolation

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1. THE RESULTS

Let w be a weight function defined on $(-1, +1)$ and let $p_n(w, \cdot)$, $n = 0, 1, 2, \ldots$, be the corresponding sequence of orthonormal polynomials with positive leading coefficient. For $f \in C[-1, +1]$ let $L_n(w, f)$ be the Lagrange interpolation polynomial of degree at most $n - 1$ coinciding with f at the zeros of $p_n(w, \cdot)$ and let $L_n^*(w, f)$ be the Lagrange interpolation polynomial of degree at most $n + 1$ coinciding with f at the zeros of the polynomial $p(t) = (t^2 - 1)\, p_n(w, t)$. Further, let $w(\alpha, \beta)$ be a Jacobi weight with $\alpha, \beta > -1$ and $\tilde{w}$ a generalized Jacobi weight as defined in [3]. If $v \geqslant 0$ is an integrable and not almost everywhere vanishing function, then for $0 < p < \infty$ a distance on the space C_v^p is given by

$$d(f, g)_{v,p} = \left[\int_{-1}^{+1} |f(t) - g(t)|^p \, v(t)\, dt \right]^{1/\max(1,p)}$$

R. Askey [1], G. P. Nevai [3] and P. Vertesi [7] have intensively studied the behavior of $d(L_n(w, f), f)_{v,p}$. Their results will be used to prove the following theorems on the behavior of $d(L_n^*(w, f), f)_{v,p}$.

THEOREM 1. (a) *Let* $w = \tilde{w}(\alpha, \beta)$ *and* $v = u \cdot w(a, b)$ *where* u^ε *is integrable for some* $\varepsilon > 1$ *and* u *is bounded in some neighborhoods of* -1 *and* $+1$. *For every* $f \in C[-1, +1]$ *we have*

$$\lim_{n \to \infty} d(L_n^*(w, f), f) = 0$$

if

$$\text{(i)} \quad -1 < \alpha, \beta \leqslant \frac{3}{2}, \quad a, b > -1, \quad p > 0,$$

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(ii) $-1 < \beta \leqslant \frac{3}{2}$, $\alpha > \frac{3}{2}$, $b > -1$, $2a > \alpha - \frac{7}{2}$,

$$0 < p < \frac{4(a+1)}{2\alpha - 3},$$

(iii) $-1 < \alpha \leqslant \frac{3}{2}$, $\beta > \frac{3}{2}$, $a > -1$, $2b > \beta - \frac{7}{2}$,

$$0 < p < \frac{4(b+1)}{2\beta - 3},$$

(iv) $\min(\alpha, \beta) > \frac{3}{2}$, $2a > \alpha - \frac{7}{2}$, $2b > \beta - \frac{7}{2}$,

$$0 < p < \min \left\{ \frac{4(a+1)}{2\alpha - 3}, \frac{4(b+1)}{2\beta - 3} \right\}.$$

(b) *Conversely, if* $\lim_{n \to \infty} d(L_n^*(w,f),f) = 0$ *for all* $f \in C_v^p$, *then* $\alpha > \frac{3}{2}$ (*resp.* $\beta > \frac{3}{2}$) *and the boundedness of* u^{-1} *in a neighborhood of* $+1$ (*resp.* -1) *implies* $p \leqslant 4(a+1)/(2\alpha - 3)$ (*resp.* $p \leqslant 4(b+1)/(2\beta - 3)$).

(c) *Further, for every weight w there exists a function* $f \in C[-1, +1]$ *and an integrable v such that for every* $p > 0$ $L_n^*(w,f)$ *does not converge to f in* C_v^p.

It would be desirable to say something about the extreme cases $p = 4(a+1)/(2\alpha - 3)$ or $p = 4(b+1)/(2\beta - 3)$. This is possible in the case $w = w(\alpha, \beta)$ and $v = w(a, b)$ with the help of

THEOREM 2. *For* $\alpha > \frac{3}{2}$ (*resp.* $\beta > \frac{3}{2}$) *and* $p \geqslant 4(a+1)/(2\alpha - 3)$ (*resp.* $p \geqslant 4(b+1)/(2\beta - 3)$) *there exists a continuous function f such that* $d(L_n^*(w,f),f)_{v,p}$ *does not converge to* 0.

Remark. A. K. Varma and P. Vertesi [6] have proved the following special cases of our Theorem 1: $w \in \{w(\frac{1}{2}, \frac{1}{2}), w(\frac{3}{2}, \frac{3}{2}), w(-\frac{1}{2}, \frac{1}{2}), w(-\frac{1}{2}, \frac{3}{2})\}$ with $v = w(-\frac{1}{2}, -\frac{1}{2})$.

2. THE PROOFS

To prove our theorems we have only to give a suitable representation of the interpolation error. Thereafter results of Nevai [3] and Vertesi [7] can easily be applied. Let $\chi_1, \chi_2, ..., \chi_n$ be the zeros of $p_n(w, \cdot)$ and let $f(\cdot, \chi_1, \chi_2, ..., \chi_n, -1, +1)$ be the $(n+2)$th divided difference with the nodes $\chi_1, \chi_2, ..., \chi_n, -1, +1$. Then by Newton's interpolation formula (see [5], p. 22) the error can be represented by

$$L_n^*(w,f,t)-f(t)=f(t,\chi_1,\chi_2,\ldots,\chi_n,-1,+1)\cdot(t^2-1)\cdot\prod_{i=1}^{n}(t-\chi_i). \quad (1)$$

Put $f^*=f(\cdot,-1,+1)=f_1-f_2$ with

$$f_1(t)=\frac{1}{2}\frac{f(t)-f(1)}{t-1} \quad \text{and} \quad f_2(t)=\frac{1}{2}\frac{f(t)-f(-1)}{t+1},$$

where $f(\cdot,-1,+1)$ is the second divided difference of f to -1 and $+1$. This immediately gives

$$d(L_n^*(w,f),f)_{v,p}=d(L_n(w,f^*),f^*)_{\tilde{v},p} \quad (2)$$

with $\tilde{v}(t)=(1-t^2)^p\cdot v(t)$, and thus Theorem 1b can easily be proven by application of [3, Theorem 2(i)].

To prove Theorem 1c we start with the note that the statement there can be generalized to any sequence (L_n) of bounded linear projection operators. Suppose that for every $f\in C[-1,+1]$ and integrable v there is a $p>0$ with

$$\lim_{n\to\infty} d(L_n(f),f)=0$$

which by the Banach–Steinhaus theorem is equivalent to the uniform boundedness of

$$\sup_{\|v\|_1=1}\int_{-1}^{1}|L_n(f,t)|^p\,v(t)\,dt=\left\|\,|L_n(f)|^p\right\|_C,$$

but this is a contradiction to the well known Harsiladze–Lozinski theorem. Hence for the proof of Theorem 1c put $L_n=L_n^*(w,\cdot)$.

Theorem 2 is proven with the help of the theorem in [7]. To prove Theorem 1a we will need the following

THEOREM [4, Theorem 9.25, p. 168]. *Let $1\leqslant p<\infty$ and let P be a polynomial of degree $m\leqslant const\cdot n$. If $w\sim w(\alpha,\beta)$ (that means $0<c_1\leqslant w/(w(\alpha,\beta))\leqslant c_2$) and $u(t)=(1-t)^\delta(1+t)^\gamma\in L_w^1$, then*

$$\sum_{i=1}^{n}\lambda_i^G(w)\,u(\chi_i)\,|P(\chi_i)|^p\leqslant const\cdot\int_{-1}^{+1}|P(t)|^p\,u(t)\,w(t)\,dt,$$

where $\lambda_i^G(w)$, $i=1,2,\ldots,n$ are the weights of the Gauss quadrature formula with the nodes χ_i, $i=1,2,\ldots,n$, and weight function w.

Now let $S_{n-1}(w,f)$ be the nth partial sum of the Fourier series expansion of f by the polynomials $p_k(w, \cdot)$, $k = 0, 1, 2, \ldots$. By the Banach–Steinhaus theorem and the inequality

$$d(L_n^*(w,f),f)_{v,p} \leqslant \text{const} \cdot d(L_n^*(w,f),f)_{v,1} \qquad \text{for} \quad 0 < p < 1,$$

it suffices to investigate the boundedness of $\|L_n^*(w, \cdot)\|_{v,p}$ $(p \geqslant 1)$ which by (2) and the inverse of Hölder's inequality is equivalent to

$$\sup_{\|f\|_C = 1} \sup_{\|g\|_{\tilde{v},q} = 1} \int_{-1}^{1} L_n(w, f^*, t)\, g(t)\, \tilde{v}(t)\, dt$$

$$\text{with} \quad p^{-1} + q^{-1} = 1. \qquad (3)$$

Four cases have to be considered:

(i) $\max(\alpha, \beta) \leqslant 0$,

(ii) $\alpha > 0, \beta \leqslant 0$,

(iii) $\beta > 0, \alpha \leqslant 0$,

(iv) $\min(\alpha, \beta) > 0$.

(i) In this case we use $f^* = f_1 - f_2$ and treat the integrals

$$\int_{-1}^{+1} L_n(w, f_j, t)\, g(t)\, \tilde{v}(t)\, dt \qquad (j = 1, 2).$$

Put $w_1(t) = (1 + t)\, w(t)$ and $w_2 = (1 - t)\, w(t)$. Then the integrals become

$$\int_{-1}^{+1} L_n(w, f_j, t)(1 \pm t)\, S_{n-1}\left(w_j, \frac{g\tilde{v}}{w_j}, t\right) w(t)\, dt,$$

where the positive sign belongs to $j = 1$ and the negative sign belongs to $j = 2$. The integrand is a polynomial of degree at most $2n - 1$ and thus Gauss quadrature gives

$$\sum_{i=1}^{n} \lambda_i^G(w)(1 \pm \chi_i) f_j(\chi_i)\, S_{n-1}\left(w_j, \frac{g\tilde{v}}{w_j}, \chi_i\right).$$

Now $\|f\|_C = 1$ implies $\|(1 \pm t) f_j(t)\|_C \leqslant 1$ and the quoted theorem of Nevai for $u \equiv 1$ gives the upper bound

$$\text{const} \cdot \sup_{\|G\|_C = 1} \int_{-1}^{+1} S_{n-1}\left(w_j, \frac{g\tilde{v}}{w_j}, t\right) G(t)\, w(t)\, dt. \qquad (4)$$

Putting $G_1(t) = G(t)/(1+t)$ and $G_2(t) = G(t)/(1-t)$ (4) is equal to

$$\begin{aligned} &\text{const} \cdot \sup_{\|G\|_C=1} \int_{-1}^{+1} g(t)\, S_{n-1}(w_j, G_j, t)\, \tilde{v}(t)\, dt \\ &\leqslant \text{const} \cdot \sup_{\|G\|_C=1} \|S_{n-1}(w_j, G_j)\|_{\tilde{v},p} \\ &\leqslant \text{const} \cdot \sup_{\|h\|_{\tilde{v},p}=1} \|S_{n-1}(w_j, h)\|_{\tilde{v},p}, \end{aligned}$$

but the last term is the norm of the linear operator $S_{n-1}(w_j, \cdot)$ considered as a mapping from $L^p_{\tilde{v}}$ into $L^p_{\tilde{v}}$.

(ii) Starting with (3) we must consider

$$\begin{aligned} &\int_{-1}^{+1} L_n(w, f^*, t)(1+t)\, S_{n-1}\left(w_1, \frac{g\tilde{v}}{w_1}, t\right) w(t)\, dt \\ &= \sum_{i=1}^{n} \lambda_i^G(w) f^*(\chi_i)(1-\chi_i^2)(1-\chi_i)^{-1}\, S_{n-1}\left(w_1, \frac{g\tilde{v}}{w_1}, \chi_i\right). \end{aligned}$$

We use Nevai's theorem for $u(t) = (1-t)^{-1}$ and get the upper bound

$$\text{const} \cdot \sup_{\|G\|_C=1} \int_{-1}^{+1} S_{n-1}\left(w_1, \frac{g\tilde{v}}{w_1}, t\right) G(t) \frac{w(t)}{1-t}\, dt,$$

which is equal to

$$\text{const} \cdot \sup_{\|G\|_C=1} \int_{-1}^{1} g(t)\, S_{n-1}(w_1, G^*, t)\, \tilde{v}(t)\, dt$$

with $G^*(t) = G(t)/(1-t)$ and this can be estimated as in (i).

(iii) This case has to be treated similarly to (ii).

(iv) In this case the integral in (3) becomes

$$\sum_{i=1}^{n} \lambda_i^G(w)(1-\chi_i^2)^{-1} f(\chi_i)(1-\chi_i^2)\, S_{n-1}\left(w, \frac{g\tilde{v}}{w}, \chi_i\right)$$

and with the help of Nevai's theorem for $u(t) = (1-t^2)^{-1}$ we get the upper bound

$$\text{const} \cdot \sup_{\|G\|_C=1} \int_{-1}^{+1} L_n(w, f^*, t)\, G(t) \frac{w(t)}{1-t^2}\, dt.$$

This leads to the same result as in the other three cases.

Now the proof must be finished as the proof of [3, Theorem 1] using results of V. M. Badkov [2] on the uniform boundedness of the norm of $S_n(w, \cdot): L_v^p \to L_v^p$.

3. An Application

Put $p = 1$ and $v \equiv 1$. Then it is immediately proven

THEOREM 3. *Let $Q_n^*(\alpha, \beta, \cdot)$ be the interpolatory quadrature formula, where the nodes are the zeros of $p(t) = (t^2 - 1) \cdot p_n(w(\alpha, \beta), t)$. Then*

$$\lim_{n \to \infty} Q_n^*(\alpha, \beta, f) = \int_{-1}^{+1} f(t)\, dt$$

for all $f \in C[-1, +1]$, if $\max(\alpha, \beta) < 7/2$.

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